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The Comass Norm of Wedge Product of Two Covectors*

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Abstract. The question "*Is the comass of wedge product of two covectors in orthogonal spaces equal to the product of the comass of them?*" was investigated first by Federer since 1960's and an affirmative answer was given for the cases of low degrees and codegrees by Morgan. We give the same answer when one of the covectors is well-known such as: complex line, exterior powers of the Kähler, torus, special Lagrangian and others.

Keywords: Calibration; Comass; Wedge product.

1. Introduction

Calibrations are useful tools for studying and constructing globally minimal surfaces on Riemannian manifolds. Calibrated geometries are the ones associated with calibrations. If a calibration calibrates all tangent planes of a submanifold M then M is area-minimizing in its homological class. There are some important problems that have not been solved in Calibrated geometry such as the classification of calibrations, characterize the boundaries of varieties in a calibrated geometry, the face of the Grassmannian $G(k, \mathbb{R}^n)\ldots$ For more details about Calibrated geometries, results and open problems, the reader is referred to [2], [5], [6], [11], [12]. In this paper, we devote to one of the problems in Calibrated geometry. It is the question that was investigated first by Federer since 1960's.

"*Is the comass of wedge product of two covectors in orthogonal spaces equal to the product of the comass of them?.*"

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The question can be stated briefly in term of calibrations as following:

"Is the wedge product of two calibrations a calibration?."

If the answer is "yes", then it is sufficient to state that:

" The Cartesian product of two calibrated submanifolds is a calibrated submanifold."

The question is important because calibrated submanifolds are area-minimizing in their homological classes. Moreover, the question has a direct relation with Problem 3.7 in [2].

Let us describe the question in more details. Let $\varphi \in \bigwedge^k(\mathbb{R}^n)^*$ and $\psi \in \bigwedge^l(\mathbb{R}^m)^*$. Consider $\varphi \wedge \psi \in \bigwedge^{k+l}(\mathbb{R}^{n+m})^*$. The question asks for proving the following equality

$$\|\varphi \wedge \psi\|^* = \|\varphi\|^* \|\psi\|^*.$$

The inequality $\|\varphi\wedge\psi\|^* \geq \|\varphi\|^*\|\psi\|^*$ is trivial, but the equality has only been proved for a few cases.

The first case, when φ or ψ is simple, was proved by Federer (see [4]); the second case, when l=2 or k=2, k=l=3 and n-k=l-m=3, was treated by F. Morgan (see [10]). In [9], by constructing a special kind of calibrations, that were called simple separated calibrations by the author himself, H. X. Huan proved the equality when one of the covectors is of that one. N. D. Binh said "yes" for the case one of two covectors is torus, by using an inductive proof (see [1]).

Let F be the set of all covectors φ satisfy $\|\varphi \wedge \psi\|^* = \|\varphi\|^*\|\psi\|^*$, for every covector ψ. By the results of Federer, Morgan, Huan and Binh as mentioned above; F contains covectors that are simple, of degree two, simple separated and torus.

In this paper, we add some kinds of covectors to this list such as: complex line, exterior powers of the Kähler, torus, special Lagrangian and other ones. The case of torus forms will be reproved by a simple and general proof.

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2. Preliminaries

2.1 Comass Norm, Faces of the Grassmannian

Let $\{e_1, e_2, \ldots, e_n\}$ be the standard orthonormal basis and $\|\ \|$ denotes Euclidean norm on $\mathbb{R}^n$. The real vector space of k-covectors (or k-forms) $\bigwedge^k(\mathbb{R}^n)^*$ has an induced orthonormal basis $\{e^*_{i_1} \wedge e^*_{i_2} \wedge \ldots \wedge e^*_{i_k} \ : \ i_1 < i_2 < \ldots < i_k\}$ and a norm, also denoted by $\|\ .\ \|$.

A second norm, the comass of a k-covector $\varphi \in \bigwedge^k(\mathbb{R}^n)^*$, is defined by:

$$\|\varphi\|^* = \max\{\varphi(\xi) \ : \ \xi \in G(k, \mathbb{R}^n\},$$

where $G(k, \mathbb{R}^n)$ is the Grassmannian of all unit simple k-vectors (oriented unit k-planes in $\mathbb{R}^n$). The set of all ξ, at which φ attains its maximum

$$G(\varphi) = \{\xi \ : \ \varphi(\xi) = \|\varphi\|^*\},$$

is called the face of the Grassmannian exposed by φ.

2.2 The Kähler and Exterior Powers of the Kähler Form

Let $\mathbb{C}^n \cong \mathbb{R}^{2n}$ denotes complex Euclidean n-space with coordinates $z = (z_1, z_2, \ldots, z_n)$, where $z = x + iy$ with $x = (x_1, x_2, \ldots x_n)$ and $y = (y_1, y_2, \ldots y_n)$, and

$$\{e_1, e_2, \ldots, e_n, e_{n+1} = ie_1, e_{n+2} = ie_2, \ldots, e_{2n} = ie_n\}$$

be the real standard orthonormal basis of $\mathbb{C}^n$. The form:

$$\Omega(u, v) = \langle u, iv\rangle,$$

is called the standard Kähler form, where $\langle , \rangle = \sum_{j=1}^n dx_j^2 + dy_j^2$ denotes the standard inner product on $\mathbb{C}^n$. This is a real 2-form. Let $(,) = \sum_{j=1}^n dz_1 \otimes d\bar{z}_j$ be the standard hermitian form. Then, we have the relationship:

$$(u, v) = \langle u, v\rangle - i\Omega(u, v).$$

In terms of axis real 2-planes, Ω is expressed as follow:

$$\Omega = e_1^* \wedge e_{n+1}^* + e_2^* \wedge e_{n+2}^* + \ldots + e_n^* \wedge e_{2n}^*.$$

The following real p-form

$$\Omega_p = \frac{1}{p!}\Omega^p$$

is called an exterior power of the Kähler.

By Wirtinger's inequality ([4]), $\Omega_p(\xi) \leq 1$, $\forall p = 1, 2, \ldots n$, with equality if and only if the real $2p$-plane ξ in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ is a complex p-plane. Each Ω_p is a calibration.

2.3 The Special Lagiangian Calibrations

With notations as in the previous subsection, the real n-form

$$\varphi = \mathrm{Re}\{dz_1 \wedge dz_2 \wedge \ldots \wedge dz_n\} \in \bigwedge^n(\mathbb{C}^n)^*$$

is well-known as a special Lagrangian form.

An oriented real n-plane ζ in $\mathbb{C}^n$ is called Lagrangian if

$$iu \perp \zeta \quad \text{for all } u \in \zeta.$$

A Lagrangian plane ζ is called special Lagrangian if

$$\zeta = A\mathbb{R}^n,$$

where $A \in SU_n$ and $\mathbb{R}^n = \{z = x + iy : \ y = 0\} \subset \mathbb{C}^n$ with the standard orientation.

We have $\varphi(\zeta) \leq |\zeta|$ for all $\zeta \in G(n, \mathbb{C}^n)$, with equality if and only if ζ is special Lagrangian (see [6], Theorem 1.10).

2.4 The Complex Line Forms

Now let $\{e_1^*, e_2^*, \ldots, e_n^*, e_{n+1}^* = (ie_1)^*, e_{n+2}^* = (ie_2)^*, \ldots, e_{2n}^* = (ie_n)^*\}$ be the dual basis of $\{e_1, e_2, \ldots, e_n, e_{n+1} = ie_1, e_{n+2} = ie_2, \ldots, e_{2n} = ie_n\}$. Denote $\omega_j = e_j^* \wedge e_{n+j}^*$. For $1 \leq p \leq n-1$, for a multy-index J with $1 \leq j_1 \leq \ldots \leq j_p \leq n$, let $\omega_J = \omega_{j_1} \wedge \ldots \wedge \omega_{j_p} \in \bigwedge^{2p}(\mathbb{C}^n)^*$ and let $\varphi = \Sigma a_J \omega_J$.

φ is called a complex line form. Its comass is equal to $\max\{a_J\}$ (cf. [3], Theorem 2.2).

We can see that the Kähler and exterior powers of the Kähler are of this form.

2.5 The Torus Forms

A torus point ξ in $G(n, \mathbb{C}^n \cong \mathbb{R}^{2n})$ is of the following form:

$$\begin{aligned}\xi &= (\cos\theta_1 e_1 + \sin\theta_1 e_{n+1}) \wedge (\cos\theta_2 e_2 + \sin\theta_2 e_{n+2}) \wedge \ldots \wedge (\cos\theta_n e_n + \sin\theta_n e_{2n}) \\ &= e^{i\theta_1} e_1 \wedge e^{i\theta_2} e_2 \wedge \ldots \wedge e^{i\theta_n} e_n.\end{aligned}$$

The set of all torus points is denoted by T. The span of T is a 2^n-dimensional subspace of $\bigwedge^n(\mathbb{R}^{2n})^*$. Let $T_S^* = \bigotimes_{i=1}^m \bigwedge^1(\text{span}\{e_i, e_{n+i}\})^* \subset \bigwedge^n(\mathbb{R}^{2n})^*$ be the dual space of T. Each form $\varphi \in T_S^*$ is called a torus form.

We can see that the special Lagrangian form is torus.

The Torus Lemma of Morgan (see [3]) says that: every torus form φ has a maximum point on the Torus T; and if φ has only finitely many maxima in T, then all of its maxima lie in T.

2.6 Decomposition of a k-covector with Respect to a Given Unit Vector

Let Φ be a k-covector on $\mathbb{R}^n$ with

$$\text{span}(\Phi)^* := \{v \in \mathbb{R}^n \ : \ v \lrcorner \Phi = 0\}^{\perp} = \mathbb{R}^n,$$

and e be a unit vector on $\mathbb{R}^n$. Set $\varphi = e \lrcorner \Phi$ and $\psi = \Phi - e^* \wedge \varphi$. Then Φ has the decomposition with respect to e

$$\Phi = e^* \wedge \varphi + \psi.$$

Note that φ is a $(k-1)$-covector and ψ is a k-covector on $e^{\perp}$, the (n-1)-dimensional subspace of all vectors orthogonal to e. It is easy to see that

$$\|\Phi\|^* \geq \varphi,$$

and equality holds if and only if $e \in \operatorname{span} \xi$ for some $\xi \in G(\Phi)$ (see [7], [8]).

3. Results

3.1 Subsets of F

In this section, some old results are summarized in few lines . After that we introduce some our recent results of this problem in term of subsets of F. Below are some important ones. The first result of this problem, *"Every simple covector belongs to F"*, was given by Federer (see [4], [6]). The second, *"Every 2-covector belongs to F"*, was proved by Morgan (see [10]). In [10], Morgan also proved for the case when φ and ψ are both of degree or codegree three. Huan treated the case of simple separated forms. This case is hard to explain briefly because the definitions are long (see [9]). By an inductive proof, Binh proved that: *"Every torus form belongs to F"* (see [1]). By observing every 3-covector in $\mathbb{R}^6$ is torus, based on a result of Morgan, he got the conclusion *"Every 3-covector in* $\mathbb{R}^6$ *belongs to F"*. Independently and simultaneously, we proved this result by a simple proof. This proof have been introduced in [12] later. We would like to show it again here because of its simplicity.

Theorem 3.1. *Let* $\varphi \in \bigwedge^3(\mathbb{R}^6)^*$ *and* $\psi \in \bigwedge^k(\mathbb{R}^n)^*$. *Then we have:*

$$\|\varphi \wedge \psi\|^* = \|\varphi\|^* \|\psi\|^*.$$

Proof. The case of $k = 1$ is trivial.

The case of $k = 2, 3$, holds by results of F. Morgan [10].

If $k = 4$, let $\xi \in G(\varphi \wedge \psi)$, then we have $\operatorname{span}\xi \cap \mathbb{R}^n \neq \{0\}$. Therefore, there exits a unit vector $e \in \operatorname{span}\xi \cap \mathbb{R}^n$, and we have:

$$\|\varphi \wedge \psi\|^* = \|e \lrcorner \varphi \wedge \psi\|^* = \|\varphi \wedge (e \lrcorner \psi)\|^* = \|\varphi\|^* \|e \lrcorner \psi\|^* \leq \|\varphi\|^* \|\psi\|^*,$$

and hence

$$\|\varphi \wedge \psi\|^* = \|\varphi\|^* \|\psi\|^*.$$

The cases of $k > 4$ can be are proved by induction. ∎

Consider a k-covector of the following form:

$$e_1^* \wedge e_2^* \wedge \varphi + \psi,$$

where φ and ψ are covectors on $\{e_1, e_2\}^\perp$. Lemma 2.1 in [3] shows that

$$\|e_1^* \wedge e_2^* \wedge \varphi + \psi\|^* = \max\{\|\varphi\|^*, \|\psi\|^*\}.$$

By using this result, we get the following:

Theorem 3.2. *Let φ be a (k-2)-covector and ψ be a k-covector on a subspace V of $\mathbb{R}^n$. If $\varphi, \psi \in F$. Then for each simple 2-covector e_{12}^* on $V^\perp$, k-covector $e_{12}^* \wedge \varphi + \psi$ belongs to F.*

Proof. Let Φ be a l-covector on $\mathbb{R}^m$. Consider $(e_{12}^* \wedge \varphi + \psi) \wedge \Phi \in \bigwedge^{k+l} \mathbb{R}^{n+m}$. Since φ and ψ belong to F, we have:

$$\begin{aligned}
\|(e_{12}^* \wedge \varphi + \psi) \wedge \Phi\|^* &= \|(e_{12}^* \wedge (\varphi \wedge \Phi) + \psi \wedge \Phi\|^* \\
&= \max\{\|\varphi \wedge \Phi\|^*, \|\psi \wedge \Phi\|^*\} \\
&= \max\{\|\varphi\|^* \|\Phi\|^*, \|\psi\|^* \|\Phi\|^*\} \\
&= \max\{\|\varphi\|^*, \|\psi\|^*\} \|\Phi\|^* \\
&= \|e_{12}^* \wedge \varphi + \psi\|^* \|\Phi\|^*.
\end{aligned}$$

The Theorem is proved. ∎

Corollary 3.3. *F contains all complex line forms. Especially, F contains all exterior powers of the Kähler form.*

Proof. We can see that, each complex line form φ can be expressed as following:

$$\varphi = \omega_1 \wedge \varphi_1 + \varphi_2,$$

where both φ_1 and φ_2 are complex line forms. Therefore, the proof of the corollary follows immediately from Theorem 3.2 by induction.

The second statement holds, since every exterior power of the Kähler form is complex line. ∎

Consider subspace $\bigwedge^1(\mathbb{R}^2)^* \otimes \bigwedge^{k-1}(\mathbb{R}^{n-2})^* \subset \bigwedge^k(\mathbb{R}^n)^*$, and let $\{e_1, e_2\}$ be an orthonormal basis of $\mathbb{R}^2$. Then every $\varphi \in \bigwedge^1(\mathbb{R}^2)^* \otimes \bigwedge^{k-1}(\mathbb{R}^{n-2})^*$ can be expressed in the following form:

$$\varphi = e_1^* \wedge \varphi_1 + e_2^* \wedge \varphi_2,$$

where φ_1 and φ_2 are in $\bigwedge^{k-1}(\mathbb{R}^{n-2})^*$. Lemma 4.1 in [3] implies that there exists a maximum point ξ of φ in the form $v \wedge \eta$, where $v \in \mathbb{R}^2$. Using this fact, we prove the following:

Theorem 3.4. *Let $\varphi \in \bigwedge^1(\mathbb{R}^2)^* \otimes \bigwedge^{k-1}(\mathbb{R}^{n-2})^* \subset \bigwedge^k(\mathbb{R}^n)^*$. If for every orthonormal basis $\{e_1, e_2\}$ on $\mathbb{R}^2$, $\varphi = e_1^* \wedge \varphi_1 + e_2^* \wedge \varphi_2$ and $\varphi_1, \varphi_2 \in F$, then $\varphi \in F$.*

Proof. Let $\Phi \in \bigwedge^l(\mathbb{R}^m)^*$ and consider $\varphi \wedge \Phi$. By Lemma 4.1, [3]; we can assume that

$$\varphi \wedge \Phi = (e_1^* \wedge \varphi_1 + e_2^* \wedge \varphi_2) \wedge \Phi = e_1^* \wedge (\varphi_1 \wedge \Phi) + e_2^* \wedge (\varphi_2 \wedge \Phi),$$

where e_1 belongs to span ξ for some maximum point ξ of $\varphi \wedge \Phi$. Then by observing in Subsection and the assumption φ_1 and φ_2 are in F, we have

$$\|\varphi \wedge \Phi\|^* = \|\varphi_1 \wedge \Phi\|^* = \|\varphi_1\|^* \|\Phi\|^* \leq \|\varphi\|^* \|\Phi\|^*.$$

And hence, the equality holds. ■

By virtue of Theorem 3.4 , if $\bigwedge^{k-1}(\mathbb{R}^{n-2})^* \subset F$, then $\bigwedge^1(\mathbb{R}^2)^* \otimes \bigwedge^{k-1}(\mathbb{R}^{n-2})^* \subset F$. Since $\bigwedge^2(\mathbb{R}^{n-2})^* \subset F$ (cf. Proposition 2.6 [3]) and $\bigwedge^3(\mathbb{R}^6)^* \subset F$ (Theorem 3.1), we get:

Corollary 3.5. *$\bigwedge^1(\mathbb{R}^2)^* \otimes \bigwedge^2(\mathbb{R}^{n-2})^* \subset \bigwedge^3(\mathbb{R}^n)^*$ and $\bigwedge^1(\mathbb{R}^2)^* \otimes \bigwedge^3(\mathbb{R}^6)^* \subset \bigwedge^4(\mathbb{R}^8)^*$ are subsets of F.*

Corollary 3.6. *T_S^*, the dual of the Torus T, is a subset of F. Especially, special Lagrangian calibrations belong to F.*

Proof. We can see that each torus form φ can be expressed as below:

$$\varphi = e_1 \wedge \varphi_1 + e_2 \wedge \varphi_2,$$

where both φ_1 and φ_2 are torus forms. Therefore, the proof of the Corollary follows immediately from Theorem 3.4 by induction.

The second statement holds, since a special Lagrangian form is torus. ■

3.2 Remarks

F contains $\bigwedge^1(\mathbb{R}^n)$ and $\bigwedge^2(\mathbb{R}^n)$, but whether F contains $\bigwedge^k(\mathbb{R}^n)$, $k \geq 3$ remains open. However, Morgan proved that the equality holds when both covectors are of degree or codegree three (see [10]). We now prove the following:

Let $\varphi \in \bigwedge^1(\mathbb{R}^2_1)^* \otimes \bigwedge^{k-1}(\mathbb{R}^{n-2})^* \subset \bigwedge^k(\mathbb{R}^n)^*$ and $\psi \in \bigwedge^1(\mathbb{R}^2_2)^* \otimes \bigwedge^{l-1}(\mathbb{R}^{m-2})^* \subset \bigwedge^l(\mathbb{R}^m)^*$. Consider $\varphi \wedge \psi \in \bigwedge^{k+l}(\mathbb{R}^{n+m})^*$. Suppose φ and ψ are of degree at most four. Then, we claim that:

$$\|\varphi \wedge \psi\|^* = \|\varphi\|^*.\|\psi\|^*.$$

Indeed, since $\varphi \wedge \psi \in \bigwedge^1(\mathbb{R}^2_1)^* \bigwedge^{k+l-1}(\mathbb{R}^{n+m-2})^*$, we have

$$\|\varphi \wedge \psi\|^* = \|e \lrcorner\, (\varphi \wedge \psi)\|^* = \|(e \lrcorner\, \varphi) \wedge \psi)\|^*,$$

where $e \in \xi$ for some $\xi \in G(\varphi \wedge \psi)$. Again, because $(e \lrcorner\, \varphi) \wedge \psi \in \bigwedge^1(\mathbb{R}^2_2)^* \bigwedge^{k+l-2}(\mathbb{R}^{n+m-4})^*$, we have

$$\|(e \lrcorner\, \varphi) \wedge \psi\|^* = \|f \lrcorner\, ((e \lrcorner\, \varphi) \wedge \psi)\|^* = \|(e \lrcorner\, \varphi) \wedge (f \lrcorner\, \psi)\|^*,$$

where $f \in \eta$ for some $\eta \in G((e \lrcorner\, \varphi) \wedge \psi)$.

Finally, since $e \lrcorner\, \varphi$ and $f \lrcorner\, \psi$ are of degree at most three, we get

$$\|(e \lrcorner\, \varphi) \wedge (f \lrcorner\, \psi)\|^* = \|e \lrcorner\, \varphi\|^* \|f \lrcorner\, \psi\|^* \leq \|\varphi\|^* \|\psi\|^*.$$

And the equality holds.

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