On EF-Extending Modules*

Nguyen Chien
Phan Chu Trinh High School, Danang city, Vietnam

Le Van Thuyet
Department of Mathematics, Hue University, 3 Le Loi Street, Hue city, Vietnam
E-mail: sciuni@png.vnn.vn

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Abstract. A module is called extending if every its submodule is essential in a direct summand. In [14], Thuyet and Wisbauer considered the extending property for the class of (essentially) finitely generated submodules. A module M is called ef-extending if every closed submodule which contains essentially a finitely generated submodule is a direct summand of M. The following implications are obvious

extending $\Rightarrow$ ef-extending $\Rightarrow$ uniform-extending

In this paper, we give an example proving that the converse is generally not true. Also, we continue the study the properties of this type of modules, especially some additional conditions for ef-extending being extending and in addition, we consider the decompositions of ef-extending and uniform-extending modules.

Keywords: extending module, ef-extending module, uniform extending module, local direct summand, relative injective, nearly injective module.

1. Preliminaries

Throughout this paper, $R$ is always an associative ring with unit and $\text{Mod-}R$ denotes the category of unital right $R$-modules. A module $M$ is called to have finite uniform dimension if $M$ does not contain an infinite direct sum of non-zero submodules. A submodule $K$ of $M$ is essential in $M$ if $K \cap L \neq 0$ for every non-zero submodule $L$ of $M$. In this case, $M$ is called an essential extension of $K$ and we write $K \leq M$. A submodule $C$ of $M$ is closed in $M$ if $C$ has no proper essential extension in $M$. A module $M$ is called extending provided every closed submodule of $M$ is a direct summand of $M$, or equivalently, every submodule of $M$ is essential in a direct summand of $M$. A module $M$ is called

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uniform-extending if every uniform submodule is essential in a direct summand of $M$. Recall that, a module which has an essential finitely generated submodule is said to be essentially finitely generated.

The texts by Anderson and Fuller [1], Dung, Huynh, Smith and Wisbauer [5], Mohamed and Müller [10], Wisbauer [17] are the general references for notions of ring and module not defined in this paper.

The following example shows that the implication ef-extending $\Rightarrow$ extending is not true.

Example 1.1. The $\mathbb{Z}$-module $M = \prod_{i=1}^{\infty} \mathbb{Z}_2$ is ef-extending but it is not extending.

Proof. It is easy to see that $N = \bigoplus_{i=1}^{\infty} \mathbb{Z}_2$ is a local direct summand of $M$. Since $\mathbb{Z}$ is a noetherian ring, $N$ is a closed submodule of $M$ (see [5, 8.1]). But $N$ is not a direct summand of $M$. In fact, suppose that $M = N \oplus K$. Set $x = (0,1,1,...,1,...) \in K$, $x' = (0,0,0,1,...,1,...) \in K$. Then $x - x' = (0,1,1,0,...,0,...) \in K \cap N$, a contradiction. Thus $M$ is not extending. We now show that $M$ is ef-extending. Since $\mathbb{Z}/2\mathbb{Z} = \{0,1\}$, $M$ has some of the following properties:

(*) Since $x = (x_i) \in M$, $x_i = 0$ or $x_i = 1$. This implies that $xk = 0$ if $k$ is even and $xk = x$ if $k$ is odd. Hence $x\mathbb{Z} = \{0,x\}$. This means that $x\mathbb{Z}$ is a simple submodule of $M$.

(**) For every $x \in M$, $x\mathbb{Z}$ is a direct summand of $M$. In fact, we can suppose that $x \neq 0$, $x = (x_i)$. Then there exists an integer $i$ such that $x_i = 1, x_1 = 1$ says, i.e., $x = (1, x_2, x_3, ...)$. Take $N' = \{(0,y_2,y_3,...) | y_i \in \mathbb{Z}_2, i > 1\} \leq M$. We can easily see that $N' \cap x\mathbb{Z} = 0$ and $M = x\mathbb{Z} \oplus N'$.

Thus, every cyclic submodule of $M$ is a simple submodule and a direct summand of $M$. So if $K$ is an essentially finitely generated submodule, then we can easily see that $K$ is a direct summand of $M$. Hence $M$ is ef-extending. 

2. Ef-Extending and Extending Modules

Clearly, an extending module is always ef-extending but the converse is generally not true, as shown in the example above. We now consider some special cases.

Proposition 2.1. Let $M$ be an ef-extending module such that every local direct summand is a direct summand of $M$. Then $M$ is an extending module.

Proof. Let $K$ be a non-zero closed submodule of $M$. For any $0 \neq x \in K$, $xR$ is essential in a submodule $A$ of $K$ which is closed in $K$. Since $K$ is closed in $M$, $A$ is closed in $M$ and therefore $A$ is a direct summand of $M$. By Zorn's lemma, there exists a maximal local direct summand $N = \bigoplus A_i$ where each $A_i \subset K$. By hypothesis, $N$ is a direct summand of $M$, i.e., $M = N \oplus N'$ for
some submodule $N'$ of $M$, so $K = N \oplus (K \cap N')$. Assume that $K \cap N' \neq 0$. Then there exists $A \neq 0$, $A$ is a direct summand of $M$. This implies that $A$ is also a direct summand of $K \cap N'$. So $N \oplus A$ is a local direct summand of $M$, contradicting the choice of $N$. Thus $K \cap N' = 0$. This means that $K = N$. This shows that $M$ is an extending module.

By the example above, we see that the $\mathbb{Z}$ module $M = \prod_{i=1}^{\infty} \mathbb{Z}/2$ is ef-extending but not extending. Note that $N = \oplus_{i=1}^{\infty} \mathbb{Z}/2$ is a local direct summand of $M$ but it is not a direct summand of $M$.

**Lemma 2.2.** A module $M$ is uniform-extending if and only if every closed submodule $K$ of $M$ that has finite uniform dimension is a direct summand of $M$.

**Proof.** See [5,7,3].

**Proposition 2.3.** For a module $M$ over a noetherian ring, the following conditions are equivalent:

(a) $M$ is ef-extending.

(b) $M$ is f-extending.

(c) $M$ is uniform-extending.

**Proof.** Since a finitely generated module over a noetherian ring is noetherian, every finitely generated module has finite uniform dimension. By Lemma 2.2, the proposition follows.

A module $M$ is said to satisfy $(C_{11})$ if and only if for every submodule $A$ of $M$, there exists a direct summand $K$ of $M$ such that $A \cap K = 0$ and $A \oplus K \leq M$. A direct sum of modules which satisfies $(C_{11})$ also satisfies $(C_{11})$ (see [13, 2.4]).

The family $\{M_i | i \in I\}$ is called relatively injective if $M_i$ is $M_j$-injective for each $i \neq j; i, j \in I$.

In [7, Theorem 3.4], Dung gave the necessary and sufficient conditions for a uniform-extending module which is a direct sum of modules with local endomorphism rings to be an extending module. We now replace it by other conditions which are related to the relative injectivity of the family of direct sum.

**Lemma 2.4.** Let $M = \oplus_{i} M_i$ be a decomposition with all $M_i$ uniform and $\text{End}(M_i)$ local. If the family $\{M_i | i \in I\}$ is relatively injective, then there does not exist an infinite sequence of non-isomorphic monomorphisms $\{M_i \xrightarrow{f_k} M_{i_{k+1}}\}_{k \in \mathbb{N}}$ with all $i_k \in I$ distinct.

**Proof.** We include here a direct module-theoretic proof, by using an idea due to Wisbauer that which was by Dung [7, Theorem 3.4]. Suppose that there exists an infinite sequence of non-isomorphic monomorphisms

$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \ldots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} \ldots$
Let \( N_i = \{ x_i - f(x_i) | x_i \in M_i \} \). Then we can easily see that the family \( \{ N_i | i = 1, 2, \ldots \} \) is independent, so the sum \( \sum_{i=1}^{\infty} N_i \) is direct. Since each \( M_i \) is a uniform module, it satisfies \( (C_{1i}) \), so does \( \bigoplus_{i=1}^{\infty} M_i \). Therefore, there exists a direct summand \( K \) of \( \bigoplus_{i=1}^{\infty} M_i \) such that \( (\bigoplus_{i=1}^{\infty} N_i) \cap K = 0 \) and \( (\bigoplus_{i=1}^{\infty} N_i) \oplus K \) is essential in \( \bigoplus_{i=1}^{\infty} M_i \). Assume that \( K \neq 0 \). Then by [1, 12.6], there exists a \( k \in \mathcal{N} \) such that \( M_k \) is a direct summand of \( K \). The relative injectivity of the family \( \{ M_i | i = 1, 2, \ldots \} \) implies that \( M_k \) is \( \bigoplus_{i \neq k} M_i \)-injective (see [10, 1.5]). Hence, there exists \( M' \) such that \( \bigoplus_{i=1}^{\infty} N_i \leq M' \) and \( \bigoplus_{i=1}^{\infty} M_i = M' \oplus M_k \). This implies that \( N_k \) is a direct summand of \( M' \) so that \( M_k \oplus N_k \) is a direct summand of \( M \) or \( M_k \oplus N_k \) is a closed submodule of \( M \). Moreover, \( M_k \oplus N_k \) is essential in \( M_k \oplus M_{k+1} \). Hence \( M_k \oplus N_k = M_k \oplus M_{k+1} \). This implies that \( f_k \) is epimorphic, a contradiction. Therefore \( K = 0 \) and hence \( \bigoplus_{i=1}^{\infty} N_i \) is essential in \( \bigoplus_{i=1}^{\infty} M_i \). Thus \( M_1 \cap (\bigoplus_{i=1}^{\infty} N_i) \neq 0 \), so there exists \( x_1 \neq 0 \), \( x_1 = y_1 - f_1(y_1) + \ldots + y_n - f_n(y_n) \), where \( y_i \in M_i (i = 1, \ldots, n) \). This would imply that \( f_1 f_2 \ldots f_n (x_1) = 0 \), which contradicts to the fact that all \( f_i \) are monomorphic, proving our lemma. 

\textbf{Theorem 2.5.} Let \( M = \bigoplus_{i \in I} M_i \) be a decomposition with \( M_i \) uniform and \( \text{End}(M_i) \) local. Assume the family \( \{ M_i | i \in I \} \) is relatively injective. Then the following conditions are equivalent:

(a) \( M \) is extending.

(b) \( M \) is ef-extending.

(c) \( M \) is uniform-extending.

\textit{Proof.} The proof follows by Lemma 2.4 and [7, Theorem 3.4]. We have omit the details. \( \square \)

3. Decompositions of EF-Extending and Uniform-Extending Modules

It is known that a finite direct sum of extending modules which are relatively injective is also an extending module. However, we have not found the answers for ef-extending module yet.

\textbf{Lemma 3.1.} Let \( M = M_1 \oplus M_2 \) having the following property: either every closed submodule \( K \) in \( M \) with \( K \cap M_1 = 0 \) is a direct summand of \( M \), or every closed submodule \( K \) in \( M \) which is essentially finitely generated such that \( K \cap M_2 = 0 \) is a direct summand of \( M \). Then \( M \) is an ef-extending module.

\textit{Proof.} Let \( K \) be a closed submodule of \( M \) that contains essentially a finitely generated submodule \( N = x_1 R + \ldots + x_n R \). Then there exists a closed submodule \( H \) in \( K \) such that \( K \cap M_2 \) is essential in \( H \). From this, \( H \) is a closed submodule of \( M \), \( H \cap M_1 = 0 \) and then \( H \) is a direct summand of \( M \), \( M = H \oplus H' \) says. This implies that \( K = H \oplus (K \cap H') \). So \( K \cap H' \cap M_2 = 0 \). We now prove that \( K \cap H' \) is essentially finitely generated. In fact, since \( N = x_1 R + \ldots + x_n R \) is essential in \( K = H \oplus (H' \cap K) \), we have
\[x_1 = h_1 + k_1, \ldots, x_n = h_n + k_n, \text{ where } h_i \in H, \ k_i \in H' \cap K(i = 1, \ldots, n)\]. Let \( B = k_1 R + \ldots + k_n R \). Since \( N \) is essential in \( K \), \( B \) is essential in \( K \cap H' \). By hypothesis, we have \( H' \cap K \) is a direct summand of \( M \) and hence of \( H' \), i.e., \( H' = (H' \cap K) \oplus P \) for some \( P \). It follows that \( M = H \oplus (H' \cap K) \oplus P = K \oplus P \), proving our lemma.

**Proposition 3.2.** A direct sum of an extending module and an ef-extending module which are relatively injective is also an ef-extending module.

**Proof.** By Lemma 3.1 and [5,7,4].

**Lemma 3.3.** Let \( M = M_1 \oplus M_2 \) with each \( M_i \) uniform and \( \text{End}(M_i) \) local \((i = 1, 2)\). Assume \( M \) is uniform-extending. Then for any \( A \leq M_i \), every homomorphism \( f : A \rightarrow M_j \) can be extended to a homomorphism \( f' : B \rightarrow M_j \), where \( B \) is a submodule of \( M_i \) such that either \( B = M_i \) or \( B \neq M_i \) and \( f' \) is an isomorphism.

**Proof.** Assume that \( A \leq M_1 \) and \( f : A \rightarrow M_2 \) is a homomorphism. Let \( A' = \{a - f(a)\mid a \in A\} \). Then \( A' \simeq A \) is a uniform submodule of \( M_2 \). Since \( M \) is uniform-extending, \( A' \) is essential in a direct summand \( D \) of \( M \). By [1, 12, 7], either \( M = M_1 \oplus D \) or \( M = D \oplus M_2 \). Assume first that \( M = D \oplus M_2 \). Let \( p : D \oplus M_2 \rightarrow M_2 \) be the projection. Then it is easy to check the restriction of \( p \) on \( M_1 \) is an extension of \( f \). So \( p \) is the desired homomorphism. Now assume that \( M = M_1 \oplus D \). Then \( D \cap M_1 = 0 \) and clearly \( \ker f = 0 \), therefore there exists \( f^{-1} : f(A) \rightarrow A \). We can easily see that the projection \( q : M_1 \oplus D \rightarrow M_1 \) which restricts on \( M_2 \) is an extension of \( f^{-1} \) and we call this extension \( j \). Since \( f^{-1} \) is a monomorphism and \( M_2 \) is a uniform module, \( j \) is also a monomorphism. We can easily see that \( A \leq j(M_2) \). Set \( B = j(M_2) \). Then we see that \( j^{-1} : B \rightarrow M_2 \) is an extension of \( f \). So \( j^{-1} \) is the desired isomorphism.

**Definition 3.4.** A module \( A \) is called nearly \( B \)-injective if for each \( C \leq B \) and for each homomorphism \( f : C \rightarrow A \) with \( \ker f \neq 0 \), then there exists a homomorphism \( f' : B \rightarrow A \) such that it is an extension of \( f \).

The family \( \{M_i\}_{i \in I} \) of right \( R \)-modules is said to satisfy \( (A_2) \) if for any choice of \( x_n, \ x_n \in M_i \) with distinct \( i_n \in I \) such that \( r_R(y) \subseteq \bigcap_{i=1}^{\infty} r_R(x_n) \) for some \( y \in M_j \), the ascending sequence:

\[\bigcap_{n=1}^{\infty} r_R(x_n) \subseteq \bigcap_{n=2}^{\infty} r_R(x_n) \ldots\]

becomes stationary.

**Lemma 3.5.** A module \( A \) is nearly \( B \)-injective if and only if \( A \) is nearly \( xR \)-injective for each \( x \in B \).
Proof. We use the same argument as that given in [10, 1.4].

Modifying the technique of Dung [8, Lemma 2.2] in the case of extending modules, we obtain the following lemma:

Lemma 3.6. Let \( M = \bigoplus_i M_i \) be a decomposition with all \( M_i \) uniform and \( \text{End}(M_i) \) local. Assume \( M_i \oplus M_j \) is uniform-extending for each pair \( i \neq j \) in \( I \) and the family \( \{M_i| i \in I\} \) satisfies \((A_2)\). Then for each \( k \in I \), \( \bigoplus_{i \neq k} M_i \) is nearly \( M_k \)-injective.

Proof. By Lemma 3.5, it suffices to prove that \( \bigoplus_{i \neq k} M_i \) is nearly \( xR \)-injective for each \( x \in M_k \). Assume that \( A \equiv xR \) and \( f : A \rightarrow \bigoplus_{i \neq k} M_i \) is a homomorphism such that \( \ker f \neq 0 \). Define \( S = \{ r \in R|xr \in A \} \). Then it is easy to check that \( S \) is an ideal of \( R \) and \( A = xS \). For each \( i \in I \setminus \{k\} \), put \( f_i = p_i f : xS \rightarrow M_i \), where each \( p_i : \bigoplus_{i \neq k} M_i \rightarrow M_i \) is the projection. Since \( M_k \oplus M_i \) is uniform-extending, \( \ker f \neq 0 \) and by Lemma 3.3, \( f_i \) can be extended to a homomorphism \( h_i : xR \rightarrow M_i \). So we can easily see that \( h : xR \rightarrow \prod_{i \neq k} M_i \)

\[
xr \mapsto (h_i(xr))_{i \in I \setminus \{k\}}
\]

is an extension of \( f \) on \( A \). Put \( a = (a_i)_{i \in I \setminus \{k\}} = h(x) \in \prod_{i \neq k} M_i \). Clearly \( r_R(x) \subseteq r_R(a) = \bigcap_{i \neq k} r_R(a_i) \). For each element \( s \in S \), let \( I_s = \{i \in I \setminus \{k\} \) such that \( a_i s \neq 0\}. Then \( I_s \) is a finite subset of \( I \setminus \{k\} \). If \( S \) is infinite, then there exists a sequence \( (s_n)_n \subseteq \bigcup_{i \in S} I_s \) such that \( \bigcup_{n=1}^\infty I_{s_n} \) is countable. Since \( I_s \) is finite for each \( s \in S \), we can choose a sequence \( (s_n)_n \) satisfying

\[
I_{s_1} \supseteq I_{s_2} \supseteq ...
\]

and \( i_1 \in I_{s_1}, i_2 \in I_{s_2} \setminus I_{s_1}, \ldots, i_n \in I \setminus \left( \bigcup_{j=1}^{n-1} I_{s_j} \right) \). Since \( i_1 \in I_{s_1} \), it follows that \( a_{i_1} s_1 \neq 0, a_{i_2} s_1 = 0 \) for each \( i \in I \setminus I_{s_1} \). Similarly, for \( i_2 \in I_{s_2} \setminus I_{s_1} \), we have \( a_{i_2} s_1 = 0, a_{i_2} s_2 \neq 0 \), ... and finally, \( i_n \in I \setminus \left( \bigcup_{j=1}^{n-1} I_{s_j} \right) \), we have \( a_{i_n} s_1 = \ldots = a_{i_n} s_{n-1} = 0, a_{i_n} s_n = 0 \).

Thus the sequence \( (\bigcap_{n=1}^\infty r_R(a_{i_n}))_{n \in N} \) is strictly increasing, contradicting to the assumption that \( \{M_i\}_{i \in I} \) satisfies \((A_2)\). We now assume that \( \bigcup_{i \in S} I_s = \{i_1, \ldots, i_n\} \). For each \( t \in I \setminus \{i_1, \ldots, i_n\} \), \( a_t s = 0 \). This would imply \( f(xs) = (a_i s)_{i \in I \setminus \{k\}} \in \bigoplus_{i=1}^n M_i \) for each \( s \in S \). Hence \( f(A) \subseteq \bigoplus_{i=1}^n M_i \). Since each \( M_i \) is nearly \( M_k \)-injective, \( \bigoplus_{i=1}^n M_i \) is nearly \( M_k \)-injective. So there exists a homomorphism \( h' : M_k \rightarrow \bigoplus_{i=1}^n M_i \), such that \( h' \) is an extension of \( f \). The proof of our lemma is completed.

Theorem 3.7. Let \( M = \bigoplus_i M_i \) be a decomposition with all \( M_i \) uniform and \( \text{End}(M_i) \) local. Then the following conditions are equivalent:

(a) \( M \) is uniform-extending.

(b) \( M_i \oplus M_k \) is extending for each pair \( k \neq i \) in \( I \) and the family \( \{M_i| i \in I\} \) satisfies \((A_2)\).
(c) $M_i \oplus M_k$ is ef-extending for each pair $k \neq i$ in $I$ and the family \{\{M_i|i \in I\} satisfies $A_2\).

(d) $M_i \oplus M_k$ is uniform-extending for each pair $k \neq i$ in $I$ and the family \{\{M_i|i \in I\} satisfies $A_2\).

Proof. (a) $\Rightarrow$ (b). See [8, Lemma 2.3].

(b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d) are obvious.

(d) $\Rightarrow$ (a). Suppose that \{\{M_i|i \in I\} satisfies $A_2\)$ and $U$ is a uniform submodule of $M$. By Zorn’s lemma, there exists $k \in I$ such that $U \cap \bigoplus_{i \neq k} M_i = 0$. Thus, the projection $p_k : M = (\bigoplus_{i \neq k} M_i) \oplus M_k \rightarrow M_k$ restricts on $U$ is a monomorphism. Let $A = p_k(U)$ and $p : (\bigoplus_{i \neq k} M_i) \oplus M_k \rightarrow (\bigoplus_{i \neq k} M_i$ be the projection. Consider the homomorphism $h : A \rightarrow (\bigoplus_{i \neq k} M_i$, defined by $h(p_k(u)) = p(u)$ for each $u \in U$. If $h = 0$ then $U \leq M_k$ and since $A$ is closed in $M$, it follows that $U = M_k$. So $U$ is a direct summand of $M$. Now assume that $h \neq 0$. Then there exists $u \in U$ such that $h(p_k(u)) \neq 0$. Thus, we can choose $i_1, i_2, ..., i_n$ in $I \setminus \{k\}$ such that $h(p_k(u)) \in M_{i_1} \oplus ... \oplus M_{i_n}$. Put $N_1 = M_{i_1} \oplus ... \oplus M_{i_n}$ and $N_2 = (\bigoplus_{i \neq k} M_i \setminus N_1$. By Lemma 3.6, $N_2$ is nearly $M$-injective and $p_N h$ is not a monomorphism (where $p_2 : (\bigoplus_{i \neq k} M_i = N_1 \oplus N_2 \rightarrow N_2$ is the projection), it would implies that $p_N h$ can be extended to a homomorphism $h_2 : M_k \rightarrow N_2$. If for each $t = 1, 2, ..., n$, $p_t h : A \rightarrow M_{i_t}$ is not a monomorphism, then $p_N h$ can be extended to a homomorphism $h_t : M_k \rightarrow M_{i_t}$. Therefore $h$ can be extended to a homomorphism $h' : M_k \rightarrow (\bigoplus_{i \neq k} M_i$. Set $M_k^* = \{x - h'(x) | x \in M_k\}$. It is easy to see that $M = M_k^* \oplus (\bigoplus_{i \neq k} M_i$ and $U \subseteq M_k^*$. Hence $U = M_k^*$, i.e., $U$ is a direct summand of $M$. If there exists some $t$ such that $p_t h$ is isomorphic then, without loss of generality, we suppose that $p_1 h, ..., p_m h$ are monomorphic for some $m \leq n$. By Lemma 3.3, $p_N h$ can be extended to a homomorphism $f_t : B_t \rightarrow M_{i_t}$ and $f_t$ is isomorphic for each $t = 1, 2, ..., m$. We can easily see that:

(*) \[ A = \bigcap_{t=1}^m B_t. \]

(**) The family \{\{B_t|t = 1, ..., m\} is total ordered.

Thus there exists $t \in \{1, ..., m\}$ such that $A = B_t$, i.e., $f_t = p_{i_t} : A \rightarrow M_{i_t}$ is isomorphic. It follows that $p_{i_t} : U \rightarrow M_{i_t}$ is isomorphic, Hence $U$ is a direct summand of $M$ and hence $M$ is uniform-extending.

References


